6

Fuzzifications

In the first three years following the appearance of *Fuzzy Sets*, Lotfi Zadeh did not publish any further developments of his new theory. He had become chairman of the department of electrical engineering at Berkeley in 1963 and was busy with the restructuring that was going on there.

6.1 Reactions

What was probably the first publicized reaction by a scientist to Zadeh's *Fuzzy Sets* came from the USSR: In its March/April 1966 edition, the Russian journal *Engineering Cybernetics* published a two-page contribution by Vasiliy I. Loginov, who interpreted Zadeh's membership function as a "likelihood function".[1] This view of memberships as "conditional probabilities" is relatively common today for the construction of membership functions using experimental methods. However, Loginov's text is also an early attestation to the immediate objection that probability theoreticians and statisticians raised against Fuzzy Set Theory: This theory was superfluous, they said, since it did not offer any greater potential for problem solving than probability theory and statistics.[2]
Zadeh traveled to Moscow once again in 1966 to deliver a lecture about Fuzzy Set Theory before the approximately 4,000 attendees at the International Congress of Mathematicians[3], and it was predominantly Soviet scientists who

1 [324] Loginov, Probability (1966); Loginov had already submitted his manuscript to the editors of *Engineering Cybernetics* on Christmas Eve of 1965

2 "Fuzzy sets" are not mentioned anywhere in the official English translation of Loginov's article. Loginov ignored the concept of Zadeh's new theory entirely. He only used Zadeh's term "membership function" and this he attributed to probability theory. It does not appear to be a clerical or typographical error that Zadeh's article is cited in Loginov's bibliography as "Hussy Sets"

3 The invitation came about at the suggestion of Stephen Kleene. [709] R. S. interview with L. A. Zadeh (2001)

proved to be open to fuzzy sets, as they recognized the contribution the concept could make to cybernetics:

> You see, in the Soviet Union they were against cybernetics until the death of Stalin. The word cybernetics was a bad word. But after the death of Stalin things changed and cybernetics became public. That's why cybernetics was used much more frequently. Cybernetics became popular in the Soviet Union but not in the United States ... And so, what I want to say is that my first paper was presented in the Soviet Union in 1965. That was my first presentation of Fuzzy Sets. My second presentation was in 1966 in Moscow at a congress of mathematicians. They were very interested – very – they learned quickly.[4]

The situation was different in the United States, though. By far the most reactions here were very negative, and some of the criticisms stooped to personal attacks. On September 13, 1965, Zadeh, in his capacity as chairman, received a letter from within his own department of electrical engineering at the University of California-Berkeley. It read as follows:

> We disagree on a very fundamental question. There is no use in kidding ourselves in saying that it is very minor. Since the disagreement lies at the very root of the academic policies of our Department, we should try to clarify our respective positions the best we can. I would say that you have a tendency of confusing our academic objectives with those appropriate for a Couturier of World Renown. There are fashions in engineering and applied science and this is in several ways very healthy. However, academic objectives are badly distorted when they can be described as "be interested only in that which is in the highest fashion today or even better in that which will be in fashion tomorrow." Such objectives are appropriate for Ives Saint Laurent but not for a chairman of an engineering department in a university. It is after many hours of reflection that I have reached the conclusion that I would be derelict in my duty if I would not oppose the appointment of a Computer Sciences candidate to any of the Assistant Professor positions that will open to us.[5]

At the *Interdisciplinary Colloquium on Mathematics in the Behavioral Sciences* no. 3 (1966–67) in Los Angeles[6], Zadeh spoke on November 18, 1966 on the subject of *Fuzzy Sets and Concepts*. Here, too, there was criticism:

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[4] [706] R. S. interview with L. A. Zadeh (2000). See also Chap. 5

[5] Zadeh, received letter (1965), no further details available

[6] The Ukrainian-American economist Jacob Marshak (1898–1977) organized this series of meetings at the University of California, Los Angeles (UCLA) from 1960 until his death

Arthur M. Geoffrion[7], who at that time was a mathematician at the Western Management Science Institute (WMSI), said that this theory could only be used when the membership function of a fuzzy set was known exactly:

> It is applicable when there is a standardized and perfectly accurate method of measuring degree of class membership, but it seems to be considerably less applicable otherwise, as when opinion is involved.[8]

In contrast, the linguist H. P. Edmundson of the University of California, Los Angeles[9] supported Zadeh's view that the fuzzy sets were needed in many disciplines, as commentaries by specialists in the fields of psychology, economics and logic had confirmed. He himself underscored

> that fuzzy sets also arise in linguistics. In particular, in the study of semantics, attempts to formulate satisfactorily the notion of a semantic space using crisp sets have essentially failed. As a consequence, the modeling of meaning as a set of senses or the modeling of synonymy in terms of equivalence classes has proved difficult to justify either theoretically or empirically. It seems likely that the concept of fuzzy set will provide a way to account for what has been called a "semantic space" and lead to a suitable metric or pseudometric. Similarly, it also may lead to a satisfactory way to replace the strict dichotomy of sentences as grammatical or ungrammatical, by a more natural concept involving grade of membership.[10]

In addition to requests for offprints, Zadeh also received encouraging feedback; the cognitive psychologist Jerome S. Bruner[11] wrote him: "But you are certainly right that we are in need of models to represent the case in which uncertainty derives from ambiguity in criterion". Philosopher Max Black[12] also responded. He had published a paper entitled *Vagueness – An Exercise in Logical Analysis*[13] back in 1934 in which he anticipated a vague idea from Zadeh's theory, for he wrote:

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[7] Arthur Minot Geoffrion (born 1937) earned his Ph.D. at Stanford University in 1965 and is now a professor of management at UCLA's Anderson School

[8] [698] Zadeh, Discussion (1966)

[9] H. P. Edmundson has been a professor of computer sciences and mathematics since 1953. He works in the field of computational linguistics

[10] [698] Zadeh, Discussion (1966)

[11] Letter from Bruner to Zadeh dated 21 June 1967; Zadeh private archive. Jerome Bruner (born 1915), professor of psychology, Harvard (1952–1972) and Oxford (1972–1980)

[12] Max Black (1909–1988), American mathematician and philosopher. Black was born – like Zadeh – in Baku, the capital of Azerbaijan, but his family emigrated soon after his birth, first to Paris and then in 1912 to London. Black became an American citizen in 1948. He was employed at the University of Illinois at Urbana and became a professor of philosophy at Cornell University in New York in 1946

[13] [83] Black, Vagueness (1937). See also [81] Black, Concepts (1963)

> The vagueness of the word chair is typical of all terms whose application involves the use of the senses. In all such cases, "borderline cases" and "doubtful objects" are easily found to which we are unable to say either that the class name does or does not apply.[14]

He now told Zadeh:

> You were good enough to send me, some time ago, some of your recent papers on topics connected with "Fuzzy Sets". If I have not written before, the reason has not been lack of interests, but an inescapable press of other duties. Now that I have had a chance, at least, to study your work, I want to express my admiration and interest. I believe that your ingenious construction promises to provide intellectual tools of great value. In case you have not come across it, I might draw your attention to an early article of mine, entitled "Vagueness" (*Philosophy of Science*, Vol. 4, 427–455; reprinted in my book, *Language and Philosophy*, Cornell University Press, 1949). A more recent article on similar topics is "Reasoning with Loose Concepts" (*Dialogue*, Vol. 2, June 1963, 1–12). I would be happy to see offprints of any of your further publications.[15]

Zadeh also presented his Fuzzy Set Theory at the First International Conference on Man and Computer, which took place in Bordeaux from June 22 to 26, 1970.[16] The title of his paper was *Fuzzy Languages and their Relation to Human and Machine Intelligence*, and his thesis was that the difference between human and mechanical intelligence lay in the ability of the human brain – "an ability which present-day digital computers do not possess – to think and reason in imprecise, non-quantitative terms". Humans could

Fig. 6.1. R. Kalman

[14] [83] Black, Vagueness (1937), p. 434
[15] Black, Letter (1967)
[16] The First International Conference on Man and Computer was organized by the Institut de la Vie at the University of Bordeaux. The conference proceedings were not published until 1972

understand inexact instructions, whereas inputs for a computer had to be defined with precision. He suggested devising *fuzzy languages* which functioned such that commands formulated in a language like this could also be processed and carried out by future computers. In the discussion which followed, Rudolf Kalman[17], who had studied in New York when Zadeh was an assistant professor, rose to speak.[18]

> Kalman was a student of mine at Columbia University so I knew him quite well. But he never changed his position ever since. He stayed with the continuum. He never wrote anything on digital discrete, never. All of his themes were connected with continuous, except for his PhD thesis where he considered Markov processes, except for that it was continuous. And so he was a strong believer in the power of mathematics and so when I started writing something was fuzzy he was very antagonistic to all that and he is to this date.[19]

Kalman's comments were blunt; it was his opinion that fuzzy sets were lacking in any scientific importance:

> We do *talk* about fuzzy things but they are not scientific concepts. Instead, let us view the development of science as something like the following. You look at a vast mass of facts – fuzzy or not – and you would like to make some sense out of it. This is usually done through rising to a higher conceptual level, by working harder than the average person. Some people in the past have discovered certain interesting things, formulated their findings in a *non-fuzzy way*, and therefore we have progressed in science.[20]

Kalman's statement, quoted here from the conference proceedings, is practically sugarcoated compared to Zadeh's recollection of the incident.[21] Almost 30 years later, Zadeh published a different account of the course of this discussion, of which I can reprint only excerpts here[22] :

> Kalman: I would like to comment briefly on Professor Zadeh's presentation. His proposals could be severely, ferociously, even brutally

[17] Rudolf E. Kalman (born 1930), Hungarian-American electrical engineer. Professor at Stanford University (1964–1971), graduate research professor and director at the Center for Mathematical System Theory, University of Florida, Gainesville (1971–1992)

[18] For more on Rudolf Kalman, see also Chap. 1

[19] [706] R. S. interview with L. A. Zadeh (2000)

[20] [262] Kalman, Discussion (1972), p. 169

[21] [706] R. S. interview with L. A. Zadeh (2000)

[22] Zadeh used quotes from this argument to introduce the speech he delivered when he was awarded the Honda Prize in 1989. It was reprinted as part 1 of *The Birth and Evolution of Fuzzy Logic – A Personal Perspective* in the *Journal of Japan Society for Fuzzy Theory and Systems*. [705] R. S. interview with L. A. Zadeh (1999)

> criticized from a technical point of view. This would be out of place here. But a blunt question remains: Is Professor Zadeh presenting important ideas or is he indulging in wishful thinking? The most serious objection of "fuzzification" of system analysis is that lack of methods of system analysis is not the principal scientific problem in the "system" field ... No doubt Professor Zadeh's enthusiasm for fuzziness has been reinforced by the prevailing political climate in the U.S. – one of unprecedented permissiveness. "Fuzzification" is a kind of scientific permissiveness; it tends to result in socially appealing slogans unaccompanied by the discipline of hard scientific work and patient observation. I must confess that I cannot conceive of "fuzzification" as a viable alternative for the scientific method; I even believe that it is healthier to adhere to Hilbert's naïve optimism, "Wir wollen wissen, wir werden wissen"...
>
> Zadeh: ... I realize, of course, that I am challenging scientific dogma ... Now, when one attacks dogma, one must be prepared to become the object of counterattack on the part of those who believe in the status quo. Thus, I am not surprised when in reaction to my views I encounter not only enthusiasm and approbation, but also criticism and derision. Nevertheless, I believe that, in time, the concepts that I have presented will be accepted and employed in a wide variety of areas ...
>
> Kalman: ... The question, then, is whether Professor Zadeh can do better by throwing away precise reasoning and relying on fuzzy concepts and algorithms. There is no evidence that he can solve any nontrivial problem ... Professor Zadeh's fears of unjust criticism can be mitigated by recalling that the alchemists were not prosecuted for their beliefs but because they failed to produce gold ...
>
> Zadeh: I will rest on what I have said, since only time can tell whether or not my ideas may develop into an effective tool for the analysis of systems which are too complex or too ill-defined to be susceptible of analysis by conventional techniques. Conceding that I am not an unbiased arbiter, my belief is that, eventually, the answer will turn out to be in the affirmative.[23]

Since then, this argument has been settled definitively in Zadeh's favor, although it smoldered until the 1990s. When Kalman spoke of "fuzzifications", he was very likely using this word disparagingly, and Zadeh's decision to call his unsharp groups "fuzzy sets" amounted to an invitation to do so. On numerous occasions, Zadeh has explained his thinking on this issue:

> I realized from the outset that the adjective "fuzzy" will create some problems because it is both pejorative and unscientific. Nevertheless,

[23] [606] Zadeh, Evolution (1999), p. 892f

> I settled on it because it described more accurately than any other term I could think of the type of imprecision which is associated with unsharp class boundaries.[24]

Today, "fuzzification" is a commonly used term both in mathematics and in technology. This chapter deals with the first "fuzzifications" of mathematical concepts and technical systems.

6.2 Fuzzy Automata

The first dissertation about fuzzy sets was an advancement of Zadeh's thoughts on the separation problem in pattern recognition. In *Fuzzy Sets and Pattern Recognition*[25], completed in December 1967, the Chinese student Chin-Liang Chang worked with a convergence proof for perceptrons which Albert B. I. Novikoff[26] of the Stanford Research Institute had presented in April of 1962.[27] Chang examined the Novikoff algorithm and the version generalized by Mark Aizerman, Emmanuel Braverman and Lev Rozonoér[28] and he expanded them to include the separation of sets by a hyperplane for the case of fuzzy sets![29]

In August of the same year, William Go Wee[30] at Purdue University in Indiana had also submitted his dissertation *On Generalizations of Adaptive Algorithms and Application of the Fuzzy Sets Concept to Pattern Classification.* Wee had written this work under King Sun Fu[31], one of the pioneers in the field

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[24] Zadeh, Lotfi A., Autobiographical note. See interviews with Zadeh: *IEEE Spectrum* 6, 1995, p. 34; in *KI* 3, 2001; see also the website: *http://www.kuenstliche-intelligenz.de/Artikel/InterviewwithProfLotfiAZadeh.htm* or at the website of *Azerbaijan International*, winter 1994 (2.4): *http://www.azer.com/aiweb/categories/magazine/24_folder/24_articles/24_fuzzylogic.html*

[25] [121] Chang, Fuzzy (1967)

[26] Albert Boris J. Novikoff, American mathematician, earned his doctorate at Stanford and today is a professor at the Courant Institute of Mathematical Sciences at New York University

[27] Novikoff spoke at the Symposium on Mathematical Theory of Automata, which took place April 24 to 26, 1962 at the Polytechnic Institute of Brooklyn. [427] Novikoff, Convergence (1962)

[28] [15] Aizerman et al., Problem (1964), [13] Aizerman et al., Foundations (1964), [14] Aizerman et al., Method (1964), [96] Braverman, Method (1965), [85] Blaydon, Pattern (1966)

[29] [427] Novikoff, Convergence (1962)

[30] William Go Wee (born 1937), Filipino electrical engineer. Wee arrived at Purdue University in Indiana in 1963

[31] King Sun Fu (1930–1985) was the founding president of the International Association for Pattern Recognition and was a professor at the Purdue School of Electrical Engineering, West Lafayette, Indiana (1960–1985). See Fig. 6.8

Fig. 6.2. M. Aizerman

of pattern recognition.[32] Wee had applied the fuzzy sets to iterative learning procedures for pattern classification and had defined a finite *automaton* based on Zadeh's concept of the fuzzy relation as a model for learning systems:

> The proposed model represents a nonsupervised learning system if a proper performance evaluator can be selected. The decision maker operates deterministically. The learning section is a fuzzy automaton. The performance evaluator serves as an unreliable "teacher" who tries to teach the "student" to make right decisions.[33]

The fuzzy automaton representing the learning section implemented a "non-supervised" learning fuzzy algorithm and converged monotonously. Wee showed that this fuzzy algorithm could not only be used in the area of pattern classification but could also be translated to control and regulation problems. He also demonstrated that the fuzzy automaton he had defined contained the concepts of deterministic and non-deterministic automata as special cases:

> Based on the concept of fuzzy relation defined by Zadeh, a class of fuzzy automata is formulated similar to that of Mealy's definition. A fuzzy automaton behaves in a deterministic fashion. However, it has many properties similar to that of a probabilistic automaton.[34]

Working with his doctoral advisor, Wee presented his findings in the article *A Formulation of Fuzzy Automata and its Applications as a Model of Learning Systems*.[35] In it he defined the fuzzy automaton – in conformity with automata theory – as a quintuple[36]:

[32] Chin-Liang Chang, who wrote a dissertation on pattern recognition under Zadeh, had also had contact with Professor Fu. Chang expresses gratitude to Fu for the conversations they shared

[33] [551] Wee, Generalization (1967), p. 101

[34] [551] Wee, Generalization (1967), p. 88

[35] [550] Wee et al., Automata (1969)

[36] See the definitions given in Chap. 3

A (finite) *fuzzy automaton* A is a quintuple (U, V, S, f, g) with

U	as a finite, non-empty input set,
V	as a finite, non-empty output set,
S	as a finite, non-empty state set,
f	as a membership function of a fuzzy set in $S \times U \times S$ to [0,1],
g	as a membership function of a fuzzy set in $V \times U \times S$ to [0,1].

They named the functions f and g the *fuzzy transition function* and the *fuzzy output function*, respectively. In particular, there is a state transition relation here for every fuzzy input, and Wee expressed them as transition matrices. The matrix for a sequence of n inputs was thus an n-ary fuzzy relation in the product space of the n transition matrices. At this point, Wee borrowed the concept of the fuzzy relation from Zadeh[37]:

> An n-valued fuzzy relation is a fuzzy set A in the product space $X \times X \times \ldots \times X$, with the membership function $f_A(x_1, \ldots, x_n)$, where $x_i \in X$, $i = 1, \ldots, n$.

In the case of binary fuzzy relations, Zadeh had defined the composition of two fuzzy relations, as those fuzzy relations whose membership function arises from those of A and B as follows:

$$f_{A \circ B}(x, y) = \sup_{\nu} \min[f_A(x, \nu), f_B(\nu, y)].$$

Since Wee only considered finite automata, he substituted the supremum in Zadeh's *sup-min composition rule* with the maximum, and by the iterative application of this *max-min composition rule* he obtained the membership function for the n transition of the fuzzy automaton.

It was while he was still in his dissertation phase that Wee co-authored with Eugene S. Santos[38] the article *General Formation of Sequential Machines*, which was published in *Information and Control* in 1968, in order to classify the different automata. Santos generalized this approach at the same time as the so-called "maximin automata".[39] Some of the definitions of such automata as systems are shown here[40]:

[37] See Chap. 5

[38] Eugene S. Santos studied at the Mapua Institute of Technology in Manila until 1963 and afterward at Ohio State University in Columbus, where he earned the Ph.D. in 1965. In 1974, he became a professor in the department of computer sciences and information systems at Youngstown State University in Youngstown, Ohio

[39] The two papers [486] Santos, Formulation (1968) and [487] Santos, Maximin (1968) were submitted at almost the same time (25 and 26 April 1968, respectively) and were published back-to-back in the same issue

[40] In contrast to the definitions in Chaps. 3 and 4, Santos considered only automata with no outputs

A *pseudo automaton* A is a system (U, S, f, F, h) for which the following is true:

U is a finite, non-empty input set,
S is a finite, non-empty state set,
f is a function from $S \times U \times S \times T$ to [0,1], where T is a subset of the real straight line, i. e. $f(s, u, s', t) \in [0, 1]$, for all $s, s' \in S, u \in U$ und $t \in T$.
f denotes the *state transition function* of A at time t.
F is a subset of S and as such is the set of *end states.*
h is a function from $S \times T$ to [0,1] and as such is the starting *distribution* of A.

If $T = \mathbf{N}$ is the set of all natural numbers, then A is called *discrete*; if f and h are independent of T, then A is called *stationary.* Elements of U are referred to as input symbols and their finite series are called *input tapes.* The "empty tape" e has the property $xe = x = ex$ for all tapes x. The denomination U^* for the set of input tapes and $lg(x)$ for the length of a tape[41] brought Santos and Wee to their next definition:

An automaton A^* is a system (U, S, f^*, F, h), with U, S, F and h being defined as above. f^* is defined like f, except that U is replaced by U^*.

For every automaton A^*, a pseudo automaton A can be found in a natural way; in other words, f is the restriction of f^* on $S \times U \times S \times T$. Santos classified the pseudo automata by looking at various restrictions f and h: A pseudo automaton is "of the class $C(C)$" if f and h satisfy a set C of conditions. If the automaton can be obtained from a pseudo automaton by an expansion rule R that is consistent with C and that clearly expands f according to f^*, then the automaton A^* is said to be "of the class $C(C, R)$". For instance, the definition of probabilistic automata as automata of the class $C(C_p, R_p)$ appeared as follows.

Probabilistic Automata $\mathbf{C}(C_P, R_P)$
The restrictions in C_P: For all $s \in S, u \in U$ and $n \in N$, it should be true that

$$\sum_{s' \in S} f(s, u, s', n) = 1 \text{ and } \sum_{s' \in S} h(s', n) = 1.$$

The expansion rule R_P: f^* is defined by induction, $lg(u^*), u^* \in U^*$

$$f(s, e, s', n) = \begin{cases} 1 & \text{if} \quad s = s' \\ 0 & \text{if} \quad s \neq s' \end{cases}$$

$$f^*(s, u^*u, s', n) = \sum_{s'' \in S} f^*(s, u^*, s'', n) f(s'', u, s', n + lg(u^*)).^{42}$$

The definition of deterministic automata turned out to be very simple.

[41] The tape y is not a k suffix of tape x if $x = zy$ for a tape z and $lg(y) = k$
[42] [487] Santos, Maximin (1968), p. 364f

Deterministic Automata $\mathbf{C}(C_D, R_D)$
C_D contains the same conditions as C_P, but also includes the restriction that the functions f and h can assume only the two values 0 and 1. The expansion rule R_D is equal to the expansion rule R_A. With the *max-min composition rule*, Santos defined the *maximin automata* as the most general of all of the automata he had studied, and he noted that "various other interesting classes of automata are given which have never appeared in any other paper before".[43]

Maximin Automata $\mathbf{C}(C_A, R_A)$
C_A is the empty set, and so there are no restrictions on the functions f and h. The expansion rule R_A for maximin automata was defined thusly:

$$f^*(s, e, s', n) = \begin{cases} 1 & \text{if} \quad s = s' \\ 0 & \text{if} \quad s \neq s' \end{cases}$$

$$f^*(s, u^*u, s', n) = \max_{s'' \in S} \min f^*(s, u^*, s'', n) f(s'', u, s', n + lg(u^*)).^{44}$$

In his very abstract text, Santos noted here only that, with this definition, the maximin automata interestingly contained both the deterministic and the non-deterministic automata as special cases. Furthermore, the state transition function f and the starting distribution h could both be interpreted as grades of membership in fuzzy sets.

6.3 Fuzzy Algorithms

Zadeh viewed his theory of fuzzy systems as a general system theory. In 1966 he had spoken in Trieste on the topic of *The Concept of State in System Theory*[45]; three years later he wrote *Towards a Theory of Fuzzy Systems.*[46] His goal was a theory for all systems – including those that were too complex or poorly defined to be accessible to a precise analysis. Alongside the systems of the "soft" fields, the "non-soft" fields were replete with systems that were only "unsharply" defined, namely "when the complexity of a system rules out the possibility of analyzing it by conventional mathematical means, whether with or without the computers".[47] As he would also do a year later in Bordeaux, Zadeh was already pointing out here the usefulness of fuzzy sets in the computer sciences: In describing their fields of application, he enumerated

[43] [487] Santos, Maximin (1968), p. 363
[44] [487] Santos, Maximin (1968), p. 364f
[45] [659] Zadeh spoke at a NATO Advanced Study Conference about network and switching theory, which took place in Trieste from August 28 to September 12, 1966; the conference proceedings were not published until 1968. [79] Biorci, Network (1968)
[46] [687] Zadeh, ERL-Toward (1969) and [668] Zadeh, Toward (1971). Hereafter, the latter publication will be cited, although the text is identical to the earlier report
[47] [668] Zadeh, Toward (1971), p. 469f

the problems that would be solved by future computers. Alongside pattern recognition, these included traffic control systems, machine translation, information processing, neuronal networks and games like chess and checkers. We had lost sight of the fact that the class of non-trivial problems for which one could find a precise solution algorithm was very limited, he wrote. Most real problems were much too complex and thus either completely unsolvable algorithmically or – if they could be solved in principle – not arithmetically feasible. In chess, for instance, there was in principle an optimal playing strategy for each stage of the game; in reality, however, no computer was capable of sifting through the entire tree of decisions for all of the possible moves with forward and backward repetitions in order to then decide what move would be the best in each phase of the game. The set of good strategies for playing chess had fuzzy limits similar to the set of tall men – these were fuzzy sets. By far the most systems that remained to be solved were fuzzy systems, and in a footnote Zadeh remarks that the maximin automata proposed by Wee and Santos were also considered examples of fuzzy systems.[48]

"How can fuzziness be made a part of system theory?" – To answer this question, Zadeh had an ace up his sleeve: In 1969 he presented "fuzzy algorithms". With that, he had fuzzified *the* central concept of computer sciences.

> The concept in question will be called a fuzzy algorithm because it may be viewed as a generalization, through the process of fuzzification, of the conventional (nonfuzzy) conception of an algorithm.[49]

Algorithms depend upon precision. An algorithm must be completely unambiguous and error-free in order to result in a solution. The path to a solution amounts to a series of commands which must be executed in succession. Algorithms formulated mathematically or in a programming language are based on set theory. Each constant and variable is precisely defined, every function and procedure has a definition set and a value set. Each command builds upon them. Successfully running a series of commands requires that each result (*output*) of the execution of a command lies in the definition range of the following command, that it is, in other words, an element of the *input set* for the series. Not even the smallest inaccuracies may occur when defining these coordinated definition and value ranges.

Once Zadeh had fuzzified input, output and state in system theory and had thus founded a theory of fuzzy systems[50], it was obvious to him how to go about fuzzifying algorithms. The commands needed to be fuzzified and so, of course, did their relations!

> I began to see that in real life situations people think certain things. They thought like algorithms but not precisely defined algorithms.[51]

[48] [668] Zadeh, Toward (1971), p. 471, footnote 1
[49] [591] Zadeh, Algorithms (1968), p. 94
[50] See Chap. 5
[51] [709] R. S. interview with L. A. Zadeh (2001)

Inspired by this idea, he wrote an article for *Information and Control* in 1968 which uncharacteristically contained neither theorems nor proofs[52]:

> Essentially, its purpose is to introduce a basic concept which, though fuzzy rather than precise in nature, may eventually prove to be of use in a wide variety of problems relating to information processing, control, pattern recognition, system identification, artificial intelligence and, more generally, decision processes involving incomplete or uncertain data. The concept in question will be called *fuzzy algorithm* because it may be viewed as a generalization, through the process of fuzzification, of the conventional (nonfuzzy) conception of an algorithm.[53] ...
>
> To illustrate, fuzzy algorithms may contain fuzzy instructions such as:
>
> (a) "Set y *approximately equal to 10* if x is *approximately equal to 5*,"
> or
>
> (b) "If x is *large*, increase y by *several* units,"
> or
>
> (c) "If x is *large*, increase y by *several* units; if x is *small*, decrease y by *several* units; otherwise keep y unchanged."
>
> The sources of fuzziness in these instructions are fuzzy sets which are identified by their underlined names.[54]

All people function according to fuzzy algorithms in their daily life, Zadeh wrote – they use recipes for cooking, consult the instruction manual to fix a TV, follow prescriptions to treat illnesses or heed the appropriate guidance to park a car. Even though activities like this are not normally called algorithms: "For our point of view, however, they may be regarded as very crude forms of fuzzy algorithms".[55]

In January 1973, Zadeh published his very carefully devised and comprehensive *Outline of a New Approach to the Analysis of Complex Systems and Decision Processes*[56], in which he not only treated fuzzy algorithms but also integrated the other fuzzifications into a new approach that was supposed to bring about a completely new form of system analysis based on his Fuzzy Set

[52] "That paper appeared in *Information and Control* even though it is not really a mathematical paper. And the reason why it appeared there is because, again, I was on the editorial board. So it could be published quickly. And I do say it's not a mathematical paper but the idea. But then other people who were mathematicians have developed that and added more mathematical and so forth. So, my function was not that of coming up with very precise. It's just an idea. That's little bit like a composer who just hums something, a sort of orchestrazing ..." [709] R. S. interview with L. A. Zadeh (2001)

[53] [591] Zadeh, Algorithms (1968), p. 694

[54] [591] Zadeh, Algorithms (1968), p. 94f. In the original, the names were not underlined but printed in italics like they are here

[55] [591] Zadeh, Algorithms (1968), p. 95

[56] [641] Zadeh, Outline (1973)

Theory: "The approach described in this paper represents a substantial departure from the conventional quantitative techniques of system analysis."[57] This new way of going about system analysis differed from the conventional approach in three ways:

- "Linguistic variables" are used instead of or in addition to numerical variables.
- Simple relationships between variables are characterized as "fuzzy IF-THEN rules" (*fuzzy conditional statements*).
- Complex relationships are characterized as "fuzzy algorithms".

Linguistic Variables:
Zadeh defined *linguistic variables* as those variables whose values are words or terms from natural or artificial languages. For instance, "not very large", "very large" or "fat", "not fat" or "fast", "very slow" are terms of the linguistic variables *size*, *fatness* and *speed*. Zadeh represented linguistic variables as fuzzy sets whose membership functions map the linguistic terms onto a numerical scale of values. In Fig. 6.3, the linguistic variable "age" is displayed with the terms "very young," "young" and "old".

Fuzzy IF-THEN Rules:
Fuzzy IF-THEN rules are composite statements of the form IF A THEN B, where A and B are fuzzy expressions, "terms with a fuzzy meaning, e. g., 'IF John is nice to you THEN you should be kind to him,' are used routinely in everyday discourse. However, the meaning of such statements when used in communication between humans is poorly defined."[58]
In those cases when the relationships among linguistic variables are more complicated than can be represented by simple fuzzy IF-THEN rules, Zadeh proposed the fuzzy algorithms.

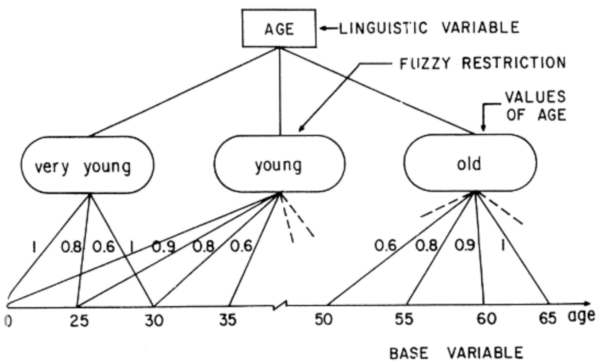

Fig. 6.3. Hierarchical structure of the linguistic variable "Age"

[57] [641] Zadeh, Outline (1973), p. 28
[58] [641] Zadeh, Outline (1973), p. 29

Fuzzy Algorithms

With reference to his earlier article about fuzzy algorithms, Zadeh provided the following definition:

> Essentially, a fuzzy algorithm is an ordered sequence of instructions (like a computer program) in which some of the instructions may contain labels of fuzzy sets, e. g.:
>
> Reduce x *slightly* if y is *large*
> Increase x *very slightly* if y is *not very large* and *not very small*
> If x is *small* then stop; otherwise increase x by 2.[59]

Zadeh divided the commands of a fuzzy algorithm into the following three classes, to which he added examples in each case:

1) *Assignment Statements*:
   $x \approx 5$
   $x =$ *small*
   x is *small*
   x is *not large* and *not very small.*

2) *Fuzzy-IF-THEN-Rules*:
   IF x is *small* THEN y is *large* ELSE y is *not large.*
   IF x is *positive* THEN decrease y *slightly.*
   IF x is *much greater* than 5 THEN stop.
   IF x is *very small* THEN go to 7.

3) *Unconditional Commands*:
   multiply x by y
   decrease x *slightly*
   delete the first *few* occurrences of 1
   go to 7
   print x
   stop.

Of course, not all commands have to be fuzzy, but they can be fuzzy; we deal most often with combinations of commands and fuzzy commands. He gave examples:

– I am requested to “take several steps”; the word “several” can then be defined as the following fuzzy set[60]: $0.5/3+0.8/4+1/5+1/6+0.8/7+0.5/8$,

or

[59] [641] Zadeh, Outline (1973), p. 30

[60] Zadeh’s notation at the time signifies in this case that the following overall numbers of steps are taken into consideration: 3, 4, 5, 6, 7 and 8, and that these numbers of steps have the following values of membership in the fuzzy set “several steps”: $\mu_{\text{several steps}}(3) = 0.5$; $\mu_{\text{several steps}}(4) = 0.8$; $\mu_{\text{several steps}}(5) = 1.0$; $\mu_{\text{several steps}}(6) = 1.0$; $\mu_{\text{several steps}}(7) = 0.8$; $\mu_{\text{several steps}}(8) = 0.5$

– Someone is told IF x is *small* THEN stop ELSE go to 7, and "small" refers to the following fuzzy set: $1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5$.

Finally, Zadeh classified the fuzzy algorithms by how they are used; these categories should be sketched out briefly.

Fuzzy Definitional Algorithms

Fuzzy sets can be defined with the aid of fuzzy definitional algorithms.

> Since a fuzzy concept may be viewed as a label for a fuzzy set, a fuzzy definitional algorithm is, in effect, a finite set of possibly fuzzy instructions which define a fuzzy set in terms of other fuzzy sets (and possibly itself, i. e. recursively) or constitute a procedure for computing the grade of membership of any element of the universe of discourse in the set under definition.[61]

To illustrate, Zadeh presented the following (simplified) fuzzy algorithm *OVAL*, which is intended to verify whether or not an object T is an oval. The word *oval* itself stood for a fuzzy concept: "The fuzzy set *oval* is the intersection of the fuzzy and nonfuzzy sets whose labels appear on the right-hand side".[62] The expression CALL *CONVEX* represents the invocation of the sub-algorithm called *CONVEX*, which is likewise a fuzzy definitional algorithm and is intended to verify whether or not T is *convex*, and finally, IF A THEN B is interpreted as follows: IF A THEN B or go to next command.

Fuzzy Algorithm OVAL

1) IF T is not closed THEN T is not *oval*; stop.
2) IF T is self-intersecting THEN T is not *oval*; stop.
3) IF T is not CALL *CONVEX* THEN T is not *oval*; stop.
4) IF T does not have two *more or less* orthogonal axes of symmetry THEN T is not *oval*; stop.
5) IF the major axis of T is not much longer than the minor axis THEN T is not *oval*; stop.
6) T is oval; stop.

Fuzzy Sub-Algorithm CONVEX[63]

1) $x = a$ (some initial point on T).
2) Choose a direction of movement along T.
3) $t \approx$ direction of tangent to T at x.
4) $x' \approx x + 1$ (move from x to a neighboring point).
5) $t' \approx$ direction of tangent to T at x'.
6) $\alpha \approx$ angle between t' and t.
7) $x \approx x'$.

---

61 [641] Zadeh, Outline (1973), p. 40

62 [641] Zadeh, Outline (1973), p. 40

63 The fuzzy sub-algorithm *CONVEX* verifies whether or not the curvature of T retains its algebraic sign if one moves along T in a randomly selected direction

8) $t \approx$ direction of tangent to T at x.
9) $x' \approx x + 1$
10) $t' \approx$ direction of tangent to T at x'.
11) $\beta \approx$ angle between t' and t.
12) IF β does not have the same sign as α THEN T is convex; return.
13) IF $x' \approx a$ THEN T is convex; return.
14) Go to 7).

Fuzzy Generational Algorithms

These fuzzy algorithms generate fuzzy sets; in Zadeh's example below, the letter "P" is generated with the height h and the base b and, for the sake of simplicity, as a dot pattern consisting of eight dots:

Fuzzy-Algorithm P(h, b)

1) $i = 1$.
2) $x(i) = b$ (first dot at base).
3) $x(i+1) \approx x(i) + \frac{h}{6}$ (put dot approximately $\frac{h}{6}$ units of distance above $x(i)$).
4) $i = i + 1$.
5) IF $i = 7$ THEN make right turn and go to 7).
6) Go to 3).
7) Move by $\frac{h}{6}$ units; put a dot.
8) Turn by 45°; move by $\frac{h}{6}$ units; put a dot.
10) Turn by 45°; move by $\frac{h}{6}$ units; put a dot.
11) Turn by 45°; move by $\frac{h}{6}$ units; put a dot; stop.

Zadeh borrowed a recipe program for chocolate sauce from a Fortran IV book by Robert S. Ledley[64] to use as an example of a fuzzy generational algorithm with feedback (see Fig. 6.4):

Fuzzy Relational and Behavioral Algorithms

Relationships between fuzzy variables can be described by fuzzy relational and behavioral algorithms to illustrate, for example, the behavior of complex systems approximatively. Using the given fuzzy sets *small* and *large*,

$$\begin{aligned}
\text{small} &= \{(1, \mu_{\text{small}}(1) = 1), (2, \mu_{\text{small}}(2) = 0.8), \\
&\quad (3, \mu_{\text{small}}(3) = 0.6), (4, \mu_{\text{small}}(4) = 0.4), \\
&\quad (5, \mu_{\text{small}}(5) = 0.2)\} \\
\text{large} &= \{(1, \mu_{\text{small}}(1) = 0.2), (2, \mu_{\text{small}}(2) = 0.4), \\
&\quad (3, \mu_{\text{small}}(3) = 0.6), (4, \mu_{\text{small}}(4) = 0.8), \\
&\quad (5, \mu_{\text{small}}(5) = 1)\}
\end{aligned}$$

a relation can now be described by the

[64] [309] Ledley, Fortran (1966); see also Chap. 7 for information on Robert S. Ledley

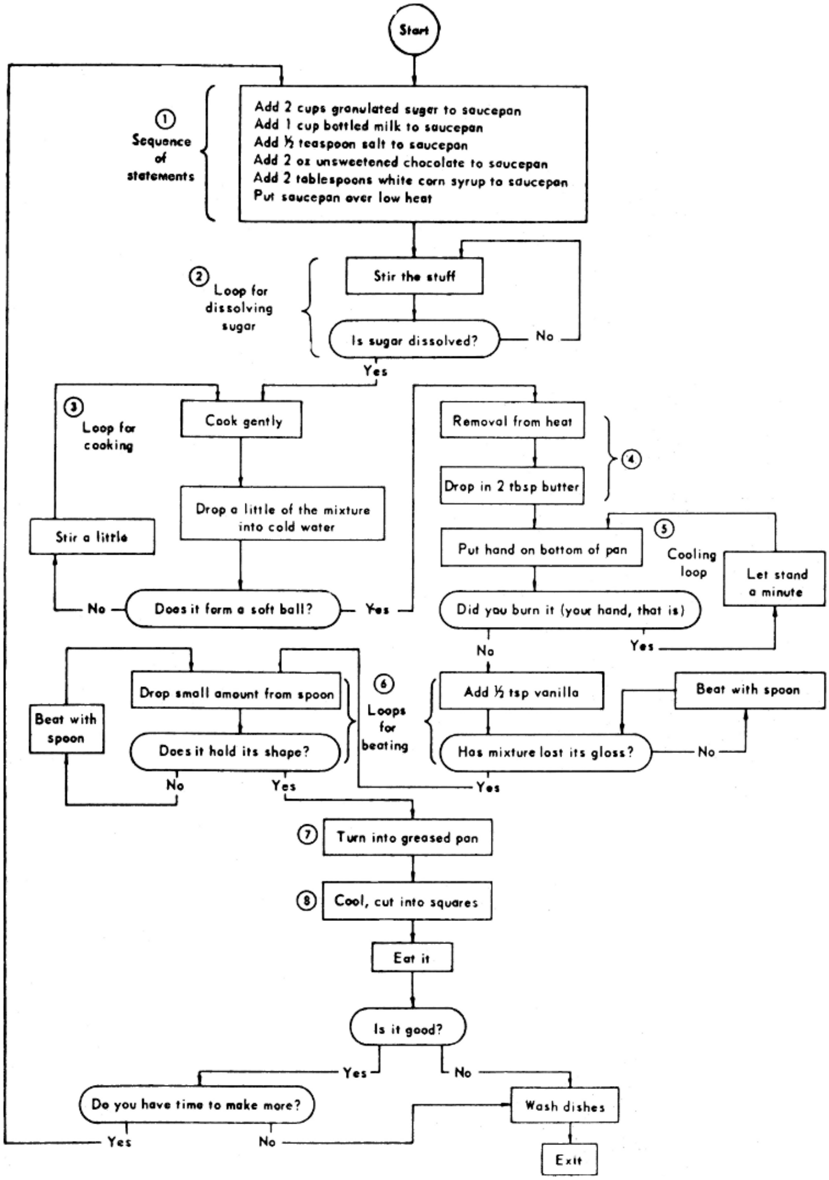

Fig. 6.4. Recipe for a chocolate sauce Zadeh adopted from R. S. Ledley

Fuzzy algorithm $\mathbf{R}(x, y, z)$:

1) IF x is *small* and y is *large* THEN z is *very small* ELSE z is *not small.*
2) IF x is *large* THEN (IF y is *small* THEN z is *very large* ELSE z is *small*) ELSE z and y are *very small.*

Using other fuzzy relational and behavioral algorithms, an attempt is made to describe system behavior approximatively by means of fuzzy relations. The relationships among inputs, states and outputs are often represented by state transition tables instead of IF-THEN rules, and this approach accordingly leads to fuzzy automata.

Fuzzy Decisional Algorithms

As examples of fuzzy decisional algorithms, Zadeh mentioned parking a car, passing through an intersection, transporting an object and buying a house. Such algorithms describe strategies or decision rules. In the example of passing through an intersection, a sub-algorithm was once again constructed for each intersection type. In his article, Zadeh considered only one of those sub-algorithms, called *SIGN*, which is responsible for the case that there is a stop signal at the intersection[65]:

Algorithm INTERSECTION
IF signal lights THEN *SIGNAL*

ELSE if stop sign THEN call *SIGN*
ELSE if blinking light THEN call *BLINKING*
ELSE call *UNCONTROLLED.*

Sub-algorithm SIGN

1) IF no stop sign on your side THEN
   IF no cars in the intersection THEN cross at *normal* speed
   ELSE wait for cars to leave the intersection and then cross.
2) IF not *close* to intersection THEN continue approaching at normal speed for a *few* seconds; go to 2).
3) *Slow down.*
4) IF in a great hurry and no police cars in sight and no cars in the intersection or its *vicinity* THEN cross the intersection at *slow* speed.
5) IF very *close* to intersection
   THEN stop; go to 7).
6) Continue *approaching* at very *slow* speed; go to 5).
7) IF no cars *approaching* or in the intersection THEN cross.
8) Wait a *few* seconds; go to 7).

[65] Zadeh often refers to the fact that he is working with simplifications here; realistic fuzzy algorithms would have to be substantially more complex in structure

6.4 Fuzzy Turing Machines

In his 1968 article *Fuzzy Algorithms for Information and Control*, Zadeh also fuzzified the Turing machine. *Fuzzy Turing machines* differ from standard Turing machines primarily by the type of dependencies the (fuzzy) state has at time $n+1$, both upon the (fuzzy) state at time n and upon the (fuzzy) input at time n. Zadeh considered a Turing machine with $Q = q_0, q_1, \ldots, q_r$ as the set of all of its states and $U = u_0, u_1, \ldots, u_m$ as the set of its tape symbols.

– In the case of a "nonfuzzy deterministic" Turing machine, the state at time $n+1$ is a function of the state at time n and of the tape symbol at time n, and therefore:

$$q^{n+1} = f(q^n, u^n),$$

wherein f is a function from $Q \times U$ to Q and q_n are variables ranging over Q and U, respectively.

– In the case of a "nonfuzzy nondeterministic" Turing machine, f is a multivalued rather than a single-valued function. The dependency can thus also more commonly be described by a relation:

$$R = \{(q^{n+1}, q^n, u^n)\}$$

wherein R is a subset of the product space $Q \times Q \times U$.

– In the case of a fuzzy Turing machine, the above relation is a fuzzy set of the product space $Q \times Q \times U$ and is characterized by the membership function $\mu_R(q^{n+1}, q^n, u^n)$, which associates with each triplet (q^{n+1}, q^n, u^n) a grade of membership in the relation R.

So a nonfuzzy nondeterministic Turing machine can be viewed as a special case of a fuzzy Turing machine whose membership function $\mu_R(q^{n+1}, q^n, u^n)$ can take only the values 1 or 0: Either (q^{n+1}, q^n, u^n) belongs to R or not. Zadeh also explained the fuzzy Turing machine as a special case of a fuzzy system. He neatly summarized his idea of understanding "fuzziness as a part of system theory" and of recommending fuzzy systems as a field of application for the computer in this way:

> In short, an algorithm corresponds to a Turing machine, a nondeterministic algorithm corresponds to a nondeterministic Turing machine, and a fuzzy algorithm corresponds to a fuzzy Turning machine.[66]

6.5 Fuzzifications of Elements of Mathematical Theory

Richard Bellman left the RAND Corporation in 1965 and was appointed Professor of Mathematics, Electrical Engineering and Medicine at the University of California, Los Angeles (UCLA).[67] This unusual title alone illustrates Bellman's multifaceted gifts and interests.

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[66] [591] Zadeh, Algorithms (1968), p. 99

[67] [709] R. S. interview with L. A. Zadeh (2001); for biographical information on Bellman, see also the *IEEE History Center* website: *http://www.ieee.org/organizations/history_center/legacies/bellman.html*

For a time, Bellman's work was largely ignored by the fraternity of mathematicians, for he was working primarily with applied mathematics – "he was a sort of excommunicated, ... he was a mathematician like Poincaré"[68] – yet he later received much acclaim for his efforts.[69]

Fig. 6.5. R. Bellman

His close friendship with Zadeh resulted in a number of co-authored publications. Bellman had been invited to deliver a lecture at the International Symposium on Multiple-Valued Logic at Indiana University in Bloomington, Indiana in May of 1975. He spoke on the subject of "Local Logics". The symposium proceedings includes only a 27-page abstract and a note indicating that the print version would be published in the book *Modern Uses in Multiple-Valued Logic.*[70] This 1977 tome does include the 62-page-long paper *Local and Fuzzy Logics* by Bellman and Zadeh[71], in which the concept of fuzzy sets is carried over to fuzzy logic. Here the authors postulate the following properties of fuzzy logic:

– Truth values here are fuzzy sets of the unit interval that has denominations like "true", "very true", "not very true", "false", "more true" or "less true", etc.

[68] [711] R. S. interview with L. A. Zadeh (2002)

[69] Among other honors, Bellman was the first to receive the *Norbert Wiener Prize in Applied Mathematics* from the American Mathematical Society and the Society for Industrial and Applied Mathematics in 1970; he was awarded the first *Dickson Price* from Carnegie-Mellon University in 1970 and the *John von Neumann Theory Award* in 1976, which was endowed jointly by the Institute of Management Sciences and the Operations Research Society of America. In 1979, Bellman was awarded the IEEE Medal of Honor, "For contributions to decision processes and control system theory, particularly the creation and application of dynamic programming." See the IEEE History Center website: *http://www.ieee.org/organizations/history_center/legacies/bellman.html*

[70] [52] Bellman, Local (1975)

[71] [51] Bellman et al., Local (1977)

– These truth values are generated by a grammar and they can be interpreted by means of semantic rules.
– Fuzzy logic is local, i. e. both the truth values and their conjunctions such as "AND", "OR" and "IF-THEN" have variable rather than fixed meanings.
– The interference rules of fuzzy logic are not exact but rather appoximative.

The concepts introduced in this article trace back to works Zadeh had published in the early '70s; they should be summarized here because they will be utilized in the next sections and the last chapter.

Fuzzy Relations

In 1971, Zadeh defined *similarity relations* and *fuzzy orderings.*[72] In doing so, he was proceeding from the concept of fuzzy relations as a fuzzification of the relation concept known in conventional set theory that he had already defined in his first text[73] on fuzzy sets[74]: If X and Y are conventional sets and if $X \times Y$ is their Cartesian product, then:

– $L(X)$ is the set of all fuzzy sets in X in X,
– $L(Y)$ is the set of all fuzzy sets in Y and
– $L(X \times Y)$ is the set of all fuzzy sets in $X \times Y$.

Relations between X and Y are subsets of its Cartesian product $X \times Y$, and the composition $t = q*r$ of the relation $q \subseteq X \times Y$ with the relation $r \subseteq Y \times Z$ into the new relation $t \subseteq X \times Z$ is given by the following definition:

$$t = q * r = \{(x, y) \exists y : (x, y) \in q \wedge (y, z) \in r\}.$$

Fuzzy relations between sets X and Y are subsets in $L(X \times Y)$. For three conventional sets X, Y and Z, the fuzzy relation Q between X and Y and the fuzzy relation R between Y and Z are defined: $Q \in L(X \times Y)$ and $R \in L(Y \times Z)$. The combination of these two fuzzy relations into a new fuzzy relation $T \in L(X \times Z)$ between X and Z can then be combined from the fuzzy relations Q and R into the new fuzzy relation $T \in L(X \times Z)$ when the logical conjunctions are replaced by the corresponding ones of the membership functions.

– The above definition of the composition of conventional relations includes a logical AND ($\wedge$) , which, for the "fuzzification", is replaced by the minimum operator that is applied to the corresponding membership functions.[75]
– The above definition of the composition of conventional relations includes the expression "$\exists y$" ("there exists a y"). The existing $y \in Y$ is the first or the second or the third ... (and so on); written logically: $(\vee) \sup\limits_{y \in Y}$. In the

[72] [654] Zadeh, Similarity (1971)
[73] [612] Zadeh, Fuzzy (1965)
[74] See Chap. 5
[75] Of course, the other proposed fuzzy operators can also be used; in those cases, correspondingly different fuzzy relations are obtained

"fuzzifications", the logical OR conjunction is replaced by the maximum operator that is applied to the corresponding membership functions.[76]

The fuzzy relation $T = Q * R$ is therefore defined via Zadeh's "combination rule of max-min combination"[77] for the membership functions:

$$\mu_T(x,y) = \max_{y \in Y} \min\{\mu_Q(x,y); \mu_r(y,z)\}, \qquad y \in Y.$$

Similarity Relations

Zadeh defined the concept of "similarity" as a generalization of the concept of the equivalence relation, since the similarity relation he defined is reflective, symmetrical and transitive, i. e. for $x, y \in X$ the membership function of S has the following properties:

(a) Reflexivity: $\mu_S(x,x) = 1$
(b) Symmetry: $\mu_S(x,y) = \mu_S(y,x)$
(c) Transitivity: $\mu_s(x,y) \geq \max_{y \in Y} \min\{\mu_S(x,y); \mu_S(y,z)\}$.

Fuzzy Ordering Relations

Fuzzy orderings are transitive fuzzy relations. The fuzzy relation P in X is a "fuzzy partial ordering" if it is reflexive, symmetrical and antisymmetrical. Antisymmetry means:

$$(\mu_P(x,y) > 0 \wedge x \neq y) \Rightarrow \mu_P(x,y) = 0.$$

Quantitative Fuzzy Semantics

Likewise in 1971, Zadeh was occupied in his paper *Quantitative Fuzzy Semantics*[78] with the definition of a language as a fuzzy relation between a set of terms $T = \{x\}$ and the universe of discourse $U = \{y\}$.
If a term x of T is given, then the membership function $\mu_L(x,y)$ defines a set $M(x)$ in U with the following membership function:

$$\mu_{M_x}(y)\mu_L(x,y).$$

Zadeh called the fuzzy set $M(x)$ *the meaning* of the term x; x is thus the name of $M(x)$. Here the following example is provided: K is the set of integers from 0 to 100, which indicate the ages of people in a population. The universe of discourse is $U = K$. The term set considered here is $T =$ *young, old, middle-aged, not old, not young, not middle-aged, young or old, not young and not old*. If we look at the term $x =$ *young*, its meaning is the fuzzy set $M($*young*$)$

[76] In addition to max operator, there are also other conjunction operations for the "fuzzy or" which then lead to other fuzzy relations
[77] The max-min composition rule is replaced in infinite sets with the sup-min composition rule. However, it is adequate to assume here that all of the sets are finite
[78] [652] Zadeh, Semantics (1971)

in U. In light of this, the following membership functionindexsrmembership function of a fuzzy set of $M(young)$ can be established subjectively:

$$\mu(y|young) = \begin{cases} 1 & \text{for} \quad y < 25 \\ \left(1 + \left(\dfrac{y-25}{5}\right)^2\right)^{-1} & \text{for} \quad y \geq 25 \end{cases}$$

Similarly, Zadeh established the membership function for $M(old)$:

$$\mu(y|old) = \begin{cases} 0 & \text{for} \quad y < 50 \\ \left(1 + \left(\dfrac{y-50}{5}\right)^{-2}\right)^{-1} & \text{for} \quad y \geq 50 \end{cases},$$

as well as the membership function for $M(middle\text{-}aged)$:

$$\mu(y|middle\text{-}aged) = \begin{cases} 0 & \text{for} \quad 0 \leq y < 35 \\ \left(1 + \left(\dfrac{y-45}{4}\right)^4\right)^{-1} & \text{for} \quad 35 \leq y < 45 \\ \left(1 + \left(\dfrac{y-45}{5}\right)^2\right)^{-1} & \text{for} \quad y \geq 45 \end{cases}.$$

Without having to go into too many details here, it should be noted that the meanings of more complex terms can also be calculated from their simpler terms by, for example, determining that "*very* x" means the square of x, and therefore "very very x" means x^4, and so forth. One illustration of this should suffice:

$x =$ *not very small* is the composite term and *small* is defined as follows (universe of discourse is $U = \{1, 2, 3, 4, 5\}$):

$$\begin{aligned} small \quad &= \quad (1, \mu_{small}(1) = 1),\ (2, \mu_{small}(2) = 0.8), \\ &\quad\ (3, \mu_{small}(3) = 0.6),\ (4, \mu_{small}(4) = 0.4), \\ &\quad\ (5, \mu_{small}(5) = 0.2) \end{aligned}$$

Therefore:

$$\begin{aligned} \textit{very small} \quad &= \quad (1, \mu_{small}(1) = 1), (2, \mu_{small}(2) = 0.64), \\ & \quad (3, \mu_{small}(3) = 0.36), (4, \mu_{small}(4) = 0.16), \\ & \quad (5, \mu_{small}(5) = 0.04). \end{aligned}$$

Using the complement formation defined for fuzzy sets, the following is also true:

$$\begin{aligned} \textit{not very small} \quad = \quad & \neg \textit{very small} \\ & (1, \mu_{small}(1) = 1), (2, \mu_{small}(2) = 0.64), \\ & (3, \mu_{small}(3) = 0.36), (4, \mu_{small}(4) = 0.16), \\ & (5, \mu_{small}(5) = 0.04). \end{aligned}$$

Zadeh later[79] defined the following "modifiers" for fuzzy sets:

- the concentration **CON** $(A) := A^2$
- the dilatation **DIL** $(A) := A^{1/2}$
- the contrast intensification

$$\mathbf{INT}(A) := \left(\begin{array}{ccc} 2A^2 & \text{for} & 0 \leq \mu_A(y) \leq 0.5 \\ \neg\, 2(\neg\, A)^2 & \text{for} & 0,5 \leq \mu_A(y) \leq 1 \end{array} \right\} .$$

The use of these modifiers on a term x occurs, for example, according to the following rules (where $A = (x, \mu_A(x))$ is the fuzzy set representing the term x):

- very x $\longrightarrow$ CON (A),
- very very x $\longrightarrow$ CON [CON (A)],
- somewhat x $\longrightarrow$ DIL (A).

6.6 Other Fuzzifications

As early as 1962, Zadeh was already trying to find "the mathematics of fuzzy or cloudy quantities" in order to be able to describe living systems.[80] At that time, even he still had no idea what this math would look like, yet seven years later he presented the ever-widening Fuzzy Set Theory at the "International Symposium on Biocybernetics of the Central Nervous System" in Boston when he spoke on the theme of *Biological Applications of the Theory of Fuzzy Sets and Systems*[81]:

[79] [641] Zadeh, Outline (1973), p. 32
[80] [662] Zadeh, System (1962), p. 857. See also Chap. 1
[81] [594] Zadeh, Biological (1969)

> The great complexity of biological systems may well prove to be an insuperable block to the achievement of a significant measure of success in the application of conventional mathematical techniques to the analysis of systems. By "conventional mathematical techniques" in this statement, we mean mathematical approaches for which we expect that precise answers to well-chosen precise questions concerning a biological system should have a high degree of relevance to its observed behavior. Indeed, the complexity of biological systems may force us to alter in radical ways our traditional approaches to the analysis of such systems. Thus, we may have to accept as unavoidable a substantial degree of fuzziness in the description of the behavior of biological systems as well as in their characterization.[82]

The program of applying fuzzy sets in the life sciences was carried out successfully in the following years, as the next chapter will show. Yet in 1969, when Zadeh spoke on this subject, there were also already scientists from these ranks who agreed with him. During the discussion which followed Zadeh's address, for example, William L. Kilmer[83] of the University of Michigan, who was working with artificial neuronal networks, developed scenarios in which Zadeh's fuzzy algorithms could be used in pattern recognition.[84]
Additionally, Zadeh found a comrade-in-arms in the fight for Fuzzy Set Theory at his own university in his colleague Hans-Joachim Bremermann[85] of the department of mathematics, for Bremermann had also recognized the great potential this theory possessed as an application in artificial neuronal networks. He spoke about this at an April 1970 meeting called *Character Recognition by Biological and Technical Systems* in Berlin.[86] *What Mathematics Can and Cannot Do for Pattern Recognition* was his subject and in his presentation he discussed the case of deformed prototypes in pattern recognition. His doctoral student Richard Hodges at Berkeley had implemented a method of measuring the degree of deformation of the prototype in letter recognition: The degree of deformation received the value 0 if a letter was identical to the

[82] [594] Zadeh, Biological (1969)
[83] William L. Kilmer earned his doctorate at the University of Michigan in 1958 and is now a professor at the Faculty of Electrical and Computer Engineering of the University of Massachusetts, Amherst
[84] [277] Kilmer, Discussion (1969)
[85] Hans-Joachim Bremermann (1926–1996), mathematician born in Bremen, Germany. Bremermann earned his doctorate in Münster in 1951 and then spent two years in the US at Stanford and Harvard. In 1959, he arrived at the mathematics department of the University of California in Berkeley and conducted research on questions of artificial intelligence. In 1979, he moved to Berkeley's department of medical physics, later the division of biophysics and cell physiology in the department of molecular and cell biology
[86] 4th Congress of the German Society for Cybernetics, held at the Technical University of Berlin on April 6–9, 1970. The proceedings were published in 1971: [207] Grüsser et al., Zeichenerkennung (1970)

prototype, but it received a small non-zero value depending on the admissible deviation from the prototype as long as the deformed prototype still belonged to the same pattern. The degree of deformation was given a large value if the deformation deviated completely from the pattern or even fell into another pattern. In Hodge's project, the deformed prototypes that were generated were modeled as fuzzy sets. The deformation function d assumed the value 0 for non-deformed prototypes, and so $(1 - d)/max\ d$ described a fuzzy set.[87] Bremermann also used this opportunity to address the criticism issued by the stochasticians, who found Fuzzy Set Theory superfluous because they did not recognize in it any innovations over probability theory:

> Fuzzy Sets have occasionally been criticized as unnecessary on the grounds that the characteristic function could or should be interpreted as a probability density. Our method shows that the critiques are wrong. It would be quite reasonable to interpret deformations as probabilities. It might be interesting to develop a *theory of groups acting on fuzzy sets*. Such a theory could accommodate deformations of a prototype as described above.[88]

Bremermann was able to inspire enthusiasm for Zadeh's fuzzy sets in yet more students. He managed to harmonize his interests in mathematics and medicine very well in the field of pattern recognition. He considered the identification of an illness as the task of adapting the system complex the patient is presenting to the known pattern of illnesses. In this way, he tied the problem of medical diagnosis into the problem of pattern recognition, which would then be implemented by the computer for processing. He had student Priscilla Wong compile the "nonfuzzy" mathematical-statistical computer diagnosis techniques that were common at the time in her Master's thesis.[89] At the same time, Merle Anne Albin wrote her dissertation under Bremermann about medical diagnostics performed with the aid of fuzzy set applications.[90]

In 1967, Bremermann had succeeded in selling the mathematician Joseph Goguen[91] on Zadeh's theory, which was still quite new at the time. *Categories of Fuzzy Sets* had been the title of Goguen's Ph.D. thesis, for which Zadeh had served as first reviewer and Bremermann as second.[92] In this work, Goguen generalized the fuzzy sets to so-called "L-sets". An L-set is a function that maps the fuzzy set carrier X into a partially ordered set L:

$$A : X \longrightarrow L.$$

[87] Bremermann noted that max d must be finite; this was not the case in Hodge's work, however

[88] [100] Bremermann, Cybernetic (1971), p. 43

[89] [572] Wong, Techniques (1975)

[90] [17] Albin, Fuzzy (1975). The subject of medical diagnostics in conjunction with Fuzzy Set Theory is reserved for the final chapter

[91] Joseph Goguen, American mathematician, is now a professor in the department of computer science and engineering at the University of California in San Diego

[92] [198] Goguen, Categories (1968)

The partially ordered set L Goguen called the "truth set" of A. The elements of L can thus be interpreted as "truth values"; in this respect, Goguen then also referred to a *Logic of Inexact Concepts*.[93] Since Zadeh's earlier definition had established this truth set as the unit interval, Fuzzy Set Theory was very soon associated with the two multi-valued logics or with probability logic, as well. Goguen's generalization of the set of values to a set L for which the only condition was to be partially ordered cleared up these misunderstandings. Goguen's work was laid out in terms of logical algebra and category theory, and his proof of a representation theorem for L-sets within category theory justified Fuzzy Set Theory as an expansion of set theory.[94]

Fig. 6.6. J. Goguen

Zadeh's efforts to use his fuzzy sets in linguistics led to an interdisciplinary scientific exchange between him and Goguen on the one hand and between the psychologist Eleanor Rosch[95] and the linguist George Lakoff[96] on the other.[97] Rosch had developed her prototype theory on the basis of empirical studies. This theory assumes that people perceive objects in the real world by comparing them to prototypes and then ordering them accordingly. In this way, according to Rosch, word meanings are formed from prototypical details and scenes and then incorporated into lexical contexts depending on the context or situation. It could therefore be assumed that different societies process perceptions differently depending on how they go about solving problems.[98] When Lakoff heard about Rosch's experiments, he was working at the Center

[93] [199] Goguen, Inexact (1969)
[94] [200] Goguen, L-fuzzy (1967). See also [199] Goguen, Inexact (1969)
[95] Eleanor Rosch (born 1938), American psychologist, professor at the University of California, Berkeley
[96] George P. Lakoff (born 1941), American linguist, Professor at the University of California, Berkeley
[97] [706] R. S. interview with L. A. Zadeh (2000), [724] R. S. interview with J. Goguen (2002)
[98] [468] Rosch, Categories (1973)

for Advanced Study in Behavioral Sciences at Stanford. During a discussion about prototype theory, someone there mentioned Zadeh's name and his idea of linking English words to membership functions and establishing fuzzy categories in this way. Lakoff and Zadeh met in 1971/72 at Stanford to discuss this idea, after which Lakoff wrote his paper *Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts.*[99] In this work, "hedges" (meaning barriers) were employed to categorize linguistic expressions. The text contained an error, which Lakoff later corrected and likewise discussed with Zadeh[100]:

> Zadeh was not interested in the points that didn't work. Zadeh was interested in the hedges and he was interested in the idea of fuzzy logic. I invented the term fuzzy logic in that paper. Goguen has used "logic of inexact concepts".[101]

Lakoff had indeed used the term "fuzzy logic" in his article and he therefore deserves the credit for first introducing this expression in the scientific literature. Based on his later research, however, Lakoff came to find that fuzzy logic was not an appropriate logic for linguistics: "It doesn't work for real natural languages, in traditional computer systems it works that way."[102]

"Inspired and influenced by many discussions with Professor G. Lakoff concerning the meaning of hedges and their interpretation in terms of fuzzy sets,"[103] Zadeh had also written an article in 1972 in which he contemplated "linguistic operators", which he called "hedges": *A Fuzzy Set-Theoretic Interpretation of Hedges.*

> A basic idea suggested in this paper in that a linguistic hedge such as *very*, *more*, *more or less*, *much*, *essentially*, *slightly* etc. may be viewed as an operator which acts on the fuzzy set representing the meaning of its operand.[104]

After this, neither Lakoff nor Zadeh made any further attempts to apply Fuzzy Set Theory to linguistics. In the mean time, however, a multitude of other developments has arisen. Once Zadeh had fuzzified the basic terms of set theory, mathematical theories were next. The first theory into which Zadeh sought to introduce his fuzzy sets was probability theory, since he had already written *Probability Measures of Fuzzy Events* during his guest residency at MIT in 1968. Any competition between probability theory and Fuzzy Set Theory he ignored completely. Instead, he suggested broadening probability theory by adding the concept of fuzzy sets:

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[99] [305] Lakoff, Hedges (1973), see also [304] Lakoff, Fuzzy (1972)
[100] [305] Lakoff, Hedges (1973), see also [304] Lakoff, Fuzzy (1972)
[101] [729] R. S. interview with G. Lakoff (2002)
[102] [729] R. S. interview with G. Lakoff (2002)
[103] [617] Zadeh, Hedges (1972), p. 4
[104] [617] Zadeh, Hedges (1972)

> By using the concept of a fuzzy set, the notions of an event and its probability can be extended in a natural fashion to fuzzy events ... It is possible that such an extension may eventually significantly enlarge the domain of applicability of probability theory, especially in those fields in which fuzziness is a pervasive phenomenon.[105]

Zadeh studied stochastic systems in a fuzzy environment in 1970 in the article *Decision-Making in a Fuzzy Environment*[106], which appeared in the journal *Management Science*. The subject here was decision-making processes in which the conditions and goals were defined as fuzzy sets. The goal function was represented as a fuzzy set of alternatives.

In the early 1970s, Richard Hamming[107] had received a letter from Japan in which the sender posed topologically motivated questions with regard to Zadeh's fuzzy sets. Hamming forwarded a copy of his reply to Zadeh:

> And then I showed this to C.-L. Chang. He was a student at that time. And he had good mathematical background and I told him, "Why don't you look into that?" And so then he looked into that and then he wrote a short paper on fuzzy topological spaces, on fuzzy topology, and that paper started this whole field with fuzzy topology.[108]

Edward T. Lee and Samuel C. Lee, two other students, co-authored the first text about *Fuzzy Neurons and Automata* in 1970.[109] This text was also published five years later in the journal *Mathematical Biosciences* under the title *Fuzzy Neural Networks*.[110] Lastly, Enrique H. Ruspini[111] should be mentioned. As a research assistant at the Brain Research Institute at UCLA in 1969, he wrote the article *A New Approach to Clustering*, in which he introduced into

105 [643] Zadeh, Probability (1968), p. 421
106 [47] Bellman, Decision (1970)
107 Richard Wesley Hamming (1915–1998), American mathematician, joined the Manhattan Project in 1945 and following the war went to Bell Telephone Laboratories, where he worked with Claude E. Shannon. In 1976, he accepted a professorship for computer science at the Naval Postgraduate School in Monterey, California
108 [706] R. S. interview with L. A. Zadeh (2000). The cited article is [122] Chang, Spaces (1968). The author in question is Chin-Liang Chang, who had also written his dissertation on pattern recognition and fuzzy sets under Zadeh in 1967; see Sect. 1 in this chapter
109 [318] Lee et al., Neurons (1970)
110 [317] Lee et al., Neural (1975)
111 Enrique Hector Ruspini (born 1942), Argentinean-American mathematician, Licenciado en Ciencias Matemáticas at the university in Buenos Aires, Argentina, Ph.D. in system science at the University of California at Los Angeles. Ruspini works at the Stanford Research Institute (SRI). He earned his doctorate in 1977 with the dissertation *A Theory of Mathematical Classification* at UCLA; third reviewer of this dissertation was Zadeh. [475] Ruspini, Theory (1977), [739] R. S. interview with E. H. Ruspini (2002). See Fig. 6.8

Fig. 6.7. E. Ruspini

the field of pattern recognition the concept of "fuzzy partition" to represent clusters in data sets:

> The object of cluster analysis is to classify experimental data in a certain number of sets where the elements of each set should be as similar as possible and dissimilar from those of the other sets. This implies the existence of a measure of distance or similarity between

Fig. 6.8. Attendees at the 1st NAFIPS Meeting in Logan, Utah, 1982. (left to right) First row: P. P. Wang, E. Mamdani, K. F. Fu, T. P. Yao, L. Saitta. Second row: M. Roubens, Ph. Smets, J. Efstathiou, R. Tong, R. R. Yager. Top row: P. Bonissone, J. Bezdek, E. H. Ruspini, E. Sanchez

> the elements to be classified. The number of such classes may be fixed beforehand or may be a consequence of some constraints imposed on them.[112]

All of these fuzzifications led to further developments, which today represent vast areas of research: fuzzy probability theory, fuzzy statistics, fuzzy algebra, fuzzy topology, fuzzy logic and many more. In 1972, Zadeh could already list a very impressive inventory.

> Since its introduction, the concept of fuzziness has been extended to algorithms, learning theory, automata, formal languages, pattern classification, probability and the decision making process.[113]

Nevertheless, it was not any of these theoretical fuzzifications that have made Zadeh's fuzzy sets popular. It was the principle of fuzzy control which proved to be very successful in the practical implementation of concepts of Fuzzy Set Theory in a small laboratory system and became the model for many other fuzzy control systems.

6.7 Fuzzy Control

In the article cited last, Zadeh and his co-author Sheldon S. L. Chang wanted to highlight an approach to fuzzifying control theory.

> There is another Chang, S. S. L. Chang, who co-authored a paper with me on fuzzy control 1972 and this is the one paper that was written mostly by S. S. L. Chang. I suggested fuzzy control to him and so he wrote something.[114]

Sheldon S. L. Chang[115] had already pursued the fuzzification of Bellman's dynamic programming in 1969.[116] Now he and Zadeh argued that control engineers tended to treat their mathematical models of physical systems as exact and precise though they knew that the models were neither.

> They obtain an optimum solution for the nominal model with possibly an added criterion of minimum sensitivity. However, no one can be sure how the system performs if it deviates from the nominal model

112 [473] Ruspini, Clustering (1969)
113 [123] Chang et al., Mapping (1972), p. 30
114 [706] R. S. interview with L. A. Zadeh (2000)
115 Sheldon S. J. Chang, Chinese-American electrical engineer. Chang studied at Tsinghua University in China and earned a doctorate at Purdue University in Lafayette, Indiana. After working several years in the industry, he went to New York University in 1952. In 1963, he became a professor at New York State University. He wrote the book *Synthesis of Optimum Control Systems*
116 [124] Chang, Programming (1969)

in some finite way. It is desirable ... to have fuzzy mathematics which represents exactly the inexact state of knowledge.[117]

To this, Chang and Zadeh added the following concepts of *fineness* and *observation*:

- The *fineness* of a fuzzy set or a fuzzy mapping represents the degree of exactitude in our knowledge about a system: The higher our standard of knowledge becomes, the greater the fineness. If we have exact knowledge about a system, the fuzzy set is a point. (The membership function has the value 1 at this point and the value 0 everywhere else.) In the case of a fuzzy mapping that represents the system, this becomes the ordinary function for exact knowledge about a system.[118]
- An *observation* is represented by an observation operator. Since our knowledge about the state increases by observation, the effect of an observation operator on a fuzzy set is to make our knowledge finer.[119]

For a state space $X = \mathbf{E}^n$ and in it the set $U \subset \mathbf{E}^n$ of all allowed controls, Chang and Zadeh defined the state of a system as the fuzzy set $p(t)$ on X. The dynamic system is then represented by a fuzzy mapping $f : X \times U \longrightarrow X$, which has the following membership function:

$$\mu_{p(t+1)}(x(t+1)) = \mu_f(x(t), u; x(t+1)).$$

They defined the observation of a fuzzy subset q of the fuzzy set p representing the state as a renormalization $\bar{q}$ of q:

$$\mu_q(x) < \mu_p(x) \qquad\qquad \mu_{\bar{q}}(x) = \frac{\mu_q(x)}{\sup\limits_{x \in X} \mu_q(x)}.$$

For the observation, Chang and Zadeh assumed an "observation operator" O. The observation operator O and the state p may be given, but the fuzzy set that represents the observed state is not unequivocal. The set of all possible $\bar{q}$ in this case was denoted by the combination $O \circ p$ and therefore $q \in O \circ p$[120]:
Finally, Chang and Zadeh defined "fuzzy *feedback control systems*":

A Fuzzy Feedback Control System consists of the following:

1) a fuzzy mapping f such that $f : X \times U \longrightarrow X$;
2) an observation operator O;
3) a goal set G on X;

[117] [123] Chang, Mapping (1972), p. 30

[118] The authors defined fuzzy mappings in this article: A fuzzy mapping f from X to Y is a fuzzy set on $X \times Y$ with membership function $\mu_f(x, y)$. A fuzzy function $f(x)$ is a fuzzy set on Y with membership function $\mu_{f(x)}(y) = \mu_f(x, y)$. ([123] Chang, Mapping (1972), p. 30)

[119] See [123] Chang, Mapping (1972), p. 30

[120] When any confusion could be ruled out, the dash over the $\bar{q}$, which denoted the observed state, was omitted

4) a control policy η which maps the observed state to a control u such that $\eta : Q \to U$, where Q is the set of observed fuzzy sets: $Q = \{q | q \in O \circ p$, p = fuzzy set on $X\}$.[121]

The fuzzy feedback control system works like this:

1) The initial state is a fuzzy set $p(0)$ on X.
2) An observation is made and the state of the system becomes $q(0) \in O$ such that $q(0) \in O \circ p(0)$.
3) $U(0) = \eta \circ q(0)$.
4) Since the fuzzy mapping, $\mu_f(x(0), u(0); x(1))$, which represents the dynamic system is given, $\mu_f(q(0), u(0); x(1))$ is obtained from the definition of a fuzzy mapping of a set.[122]
5) $p(1)$ becomes the new initial state and steps 2), 3), 4) are repeated with the increased time variable.

With that, Chang and Zadeh had represented a control problem with the triplet (f, O, G) and a feedback control problem with a triplet (f, O, η). The goal G is considered achievable if an η exists such that for a t which can potentially be determined: $q(t) \subseteq G(t)$. In this paper, Chang and Zadeh proved the following central

Theorem 3. *Let $P_1(f_1, O_1, \eta)$ and $P_2(f_2, O_2, \eta)$ be two control problems such that f_2 is finer than f_1 and O_2 permits better observation than O_1, then G is attainable in P_2 if it is attainable in P_1. For feedback control problems, this means that a precise goal can be attained with rather sloppy control and observation. However, as the goal is approached the observation must be precise, since otherwise one cannot tell whether the goal is attained or not.*

This result yielded the theoretical proof that complex systems can be described and successfully controlled with the help of fuzzy sets. The mathematics of fuzzy sets also resulted in success in cases where precise mathematics failed because the complexity of real systems could not be overcome. In the area of control theory and its technical applications, this discrepancy between theory and practice was keenly felt when mathematical assumptions did not accord with actual conditions or when it was not expedient to produce mathematically exact calculations because it would take too long to obtain the results.
With his article *A Rationale for Fuzzy Control* in the *Journal of Dynamic Systems, Measurement and Control* the same year, Zadeh campaigned among control theoreticians and engineers to employ the theory of fuzzy sets and fuzzy algorithms.[123] The advent of the age of space travel in the year

[121] [123] Chang, Mapping (1972), p. 32
[122] If A is a fuzzy set on X, then the fuzzy set $f(A)$ is defined by the following membership function: $\mu_{f(A)}(y) \equiv \sup_{x \in X} (\mu_A(x) \wedge \mu_f(x, y))$
[123] [647] Zadeh, Rationale (1972)

Fig. 6.9. S. L. Chang

1957 had had a dramatic effect on control theory, he wrote. Heretofore the level of mathematization had been relatively low and publications on control theory had only very seldom included theorems and proofs. Since 1960, however, the "level of mathematical sophistication" had grown rapidly, "swinging the pendulum all the way from the low-brow imprecision of the forties to the high-brow mathematical formalism of the seventies".[124] Nowadays, in the early 1970s, it was a "must" for any article in a high-class control engineering periodical to include at least a few theorems and proofs. This was laudable, he said, as it forced the author to make precise statements about his assumptions and conclusions, but this trend also had its disadvantages:

> On the other hand, the question for precision frequently tends to overshadow other, perhaps more important goals, such as the invention of new types of control systems or the discovery of results which, though not of mathematical interests, are of high relevance to real world problems.[125]

A visible consequence of this "excessive concern with precision" was that large portions of the literature about control theory were moving further and further away from reality and contributing less and less to solving problems involving complex and large-scale man-machine systems, which was at the root of the many crises confronting modern society.

> In short, I believe that excessive concern with precision has become a stultifying influence in control and system theory, largely because it tends to focus the research in these fields on those, and only those, problems which are susceptible of exact solutions.[126]

The host of important problems with data, objects or conditions that were too complex or too imprecisely defined to be susceptible to a mathematically

[124] [647] Zadeh, Rationale (1972), p. 3
[125] [647] Zadeh, Rationale (1972), p. 3f
[126] [647] Zadeh, Rationale (1972), p. 4

exact analysis were and are ignored due to "mathematical intractability". For problems of this type, this stubborn insistence upon precision had to be abandoned and answers permitted which were *fuzzy* or *uncertain*. Zadeh expressed the hope that his call would be heeded in the future:

> In conclusion, I believe that in the years ahead fuzzy algorithms and control policies will gain increasing though perhaps grudging acceptance. They will have to be accepted and accorded some measure of respectability because the conventional nonfuzzy algorithms cannot, in general, cope with the complexity and ill-definedness of large-scale systems. I also believe that, in order to provide a hospitable environment for the development of fuzzy algorithms, control theory must become less preoccupied with mathematical rigor and precision, and more concerned with the development of qualitative or approximate solutions to pressing real world problems. Such a theory may well turn out to be far richer and far more exciting than control theory today.[127]

It would take about another year before a development began which would prove Zadeh right once again..

6.8 The First "Fuzzy Logic Controller" for Controlling a Steam Engine

The standard methods of designing automatic control systems that were taught until the early 1970s were based on mathematical models that had been established in advance, according to which all numerical calculations were then carried out. A number of newer studies were concerned with self-organizing or adaptive control systems[128] in which the control strategy was not determined in advance but could be adapted to an optimization algorithm. In this way, the experiences of control experts could be taken into account during the optimization process. These control systems displayed several properties that bore a resemblance to human learning; they were therefore also presented as examples of "artificial intelligence" or as "learning machines".
Ebrahim Mamdani[129] at Queen Mary College in London was also interested in the so-called "learning machines". Earlier he had worked in the field of pattern

[127] [647] Zadeh, Rationale (1972), p. 4

[128] [573] Yovits et al., Systems (1962), [538] Truxal, Control (1963), [373] Mendel et al., Techniques (1970)

[129] Ebrahim H. Mamdani, Indian-British electrical engineer, studied at the College of Engineering in Poona, India and then went to England, where be became professor of electrical engineering at London's Queen Mary College and at Westfield College of London University (1984–1995). Since 1995, Mamdani has been a professor of electrical engineering at the Imperial College of Science, Technology and Medicine at the University of London. [731] R. S. interview with E. Mamdani (1998)

Fig. 6.10. E. Mamdani

recognition with artificial neuronal networks, in particular the recognition of handwriting and spoken language. Leaning and self-organization were two of the many attributes ascribed to human intelligence. Mamdani now expressed an interest in a third facet of human intelligence: the ability to comprehend commands and develop strategies on the basis of verbal communication rather than experience. Here he fell back on Alexander R. Luria's[130] graphic illustrations, which showed how the development of perceptual-motor skills in small children had a verbal basis.[131] He likewise invoked the cybernetic theory of learning devised by Andrew Gordon Speedie Pask[132], which stressed the linguistic nature of many aspects of intelligence behavior. Lastly, the artificial intelligence researcher Terry Winograd[133] had demonstrated shortly before that a robot arm could be controlled linguistically in order to manipulate various toy blocks.[134] Mamdani now wished to design a control system that could "learn" on the basis of linguistic rules. Naturally, only a fraction of a linguistic structure, such as the one Terry Winograd had shown, was to be considered: "... we were primarily concerned with the translation of semantic expressions into control laws, and not with the recognition of the expressions themselves or their manipulation. To the control engineer quantitative languages supporting arithmetic are the natural ones."[135]

Commands provided and understood by human control experts were to be learned. Normally, an expert observed the sequence of processes and knew based on experience how he should intervene if necessary. If any rules governed how he should proceed, they would include linguistically vague expressions, since he would use worlds like "much", "little", "some", "very" and so forth.

[130] Alexander Romanovich Luria (1902–1977), Russian neuropsychologist

[131] [332] Luria, Speech (1960)

[132] [437] Pask, Learning (1963), [438] Pask, Method (1971). Andrew Gordon Speedie-Pask (1928–1996), English cyberneticist. Pask was a professor in the department of cybernetics at Brunel University in England

[133] Terry Winograd (born 1946), American computer scientist. In the late 1960s, he constructed the AI program SHRDLU. Winograd is a professor of computer science at Stanford University

[134] [569] Winograd, Language (1972)

[135] [348] Mamdani, Fuzzy (1975), p. 2

In his January 1973 article *Outline of a New Approach to the Analysis of Complex Systems and Decision Processes*, Zadeh had identified words such as this as linguistic terms or modifiers of linguistic variables.

> The true antecedent of the work described here is an outstanding paper by Zadeh (1973) which lays the foundations of what we have termed linguistic synthesis ... and which had also been described by Zadeh as Approximate Reasoning (AR). In the 1973 paper Zadeh shows how vague logical statements can be used to derive inferences (also vague) from vague data. The paper suggests that this method is useful in the treatment of complex humanistic systems. However, it was realized that this method could equally be applied to "hard" systems such as industrial plant controllers.[136]

Mamdani had read this article and subsequently suggested to his doctoral student Sedrak Assilian[137] that he devise a fuzzy algorithm to control a small model steam engine (Fig. 6.11).[138]

The entire system consisted of the combination of a steam engine and a boiler (see Fig. 6.12). The steam was supposed to reach a certain predetermined pressure within the boiler; this was achieved by regulating the temperature. The engine was to run as consistently as possible at a particular piston speed, for which purpose a throttle was installed. This was therefore a system with

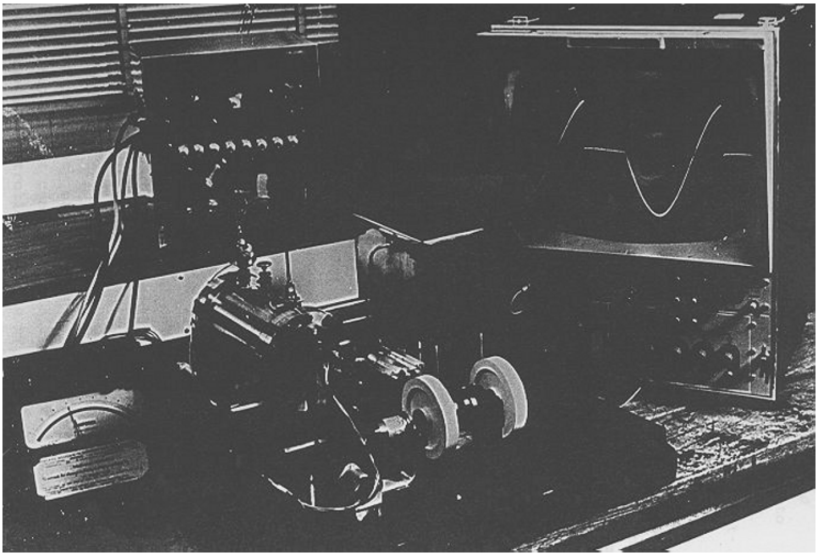

Fig. 6.11. The fuzzy-controlled steam engine

[136] [350] Mamdani, Advances (1976), p. 325
[137] No other facts about Sedrak Assilian are available; he also does not appear in later literature about Fuzzy Set Theory and its applications
[138] [731] R. S. interview with E. Mamdani (1998)

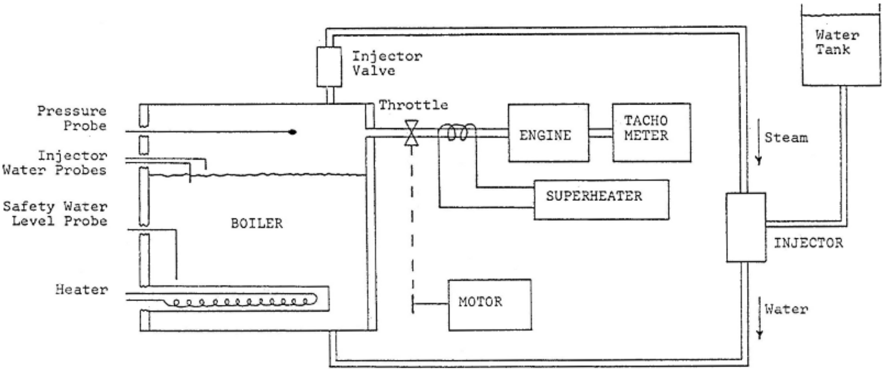

Fig. 6.12. Diagram of the system consisting of a steam engine and a boiler

two inputs (heat supplied to the boiler, engine throttle) and two outputs (pressure in the boiler, engine speed) (see Fig. 6.14).
Sensors constantly monitored the boiler and indicated the current pressure. If the prevailing pressure corresponded to the set point value, then nothing needed be done. If it deviated from the set point, then some action had to be taken, and this task was to be assumed by an automatic fuzzy controller.

> Simple identification tests on the plant proved that it is highly nonlinear with both magnitude and polarity of the input variables. Therefore the plant possesses different characteristics at different operating points, so that the direct digital controller implemented for comparison purposes had to be returned (by trial and error) to give the best performance each time the operating point was altered.[139]

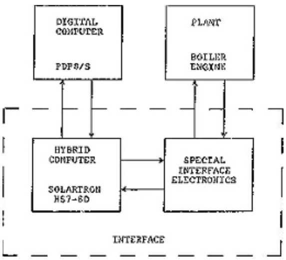

Fig. 6.13. The system of the fuzzy steam engine

[139] [348] Mamdani, Fuzzy (1975), p. 2

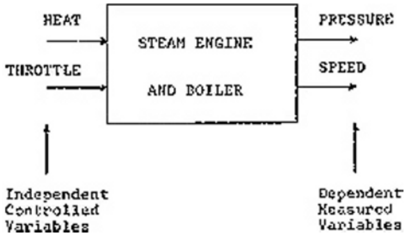

Fig. 6.14. The process variables of the fuzzy steam engine

Assilian and Mamdani defined six linguistic variables (four input and two output variables):

(1) PE *Pressure Error*, defined as the difference between the actual value and the set point of the pressure in the boiler.
(2) SE *Speed Error*, defined as the difference between the actual value and the set point of the of the piston speed.
(3) CPE *Change in pressure error*, defined as the difference between the actual value of PE and its most recent value.
(4) CSE *Change in speed error*, defined as the difference between the actual value of SE and its most recent value.
(5) HC *Heat Change* (action variable, as the result of which a command occurs).
(6) TC *Throttle Change* (action variable, as the result of which a command occurs).

Mamdani and Assilian introduced linguistic terms for these variables:

PB -	Positive, Big
PM -	Positive, Mittel
PS -	Positive, Small
$P0$ -	Positive, Zero
$N0$ -	Negative, Zero
NS -	Negative, Small
NM -	Negative, Medium
NB -	Negative, Big.

The variables were distributed over a number of points in accordance with the universe of discourse. For the variables PE and SE there were 13 points, which ranged from the maximum negative error through zero to the maximum positive error, with the zero being divided into a "negative zero error" NO and a "positive zero error" PO ("NO - just below the set point ... PO - just above the set point"[140]). Assilian defined fuzzy sets subjectively with the following values (see Fig. 6.15):

[140] [348] Mamdani, Fuzzy (1975), p. 7f

	−6	−5	−4	−3	−2	−1	−0	+0	+1	+2	+3	+4	+5	+6
PB	0	0	0	0	0	0	0	0	0	0	0·1	0·4	0·8	1·0
PM	0	0	0	0	0	0	0	0	0	0·2	0·7	1·0	0·7	0·2
PS	0	0	0	0	0	0	0	0·3	0·8	1·0	0·5	0·1	0	0
PO	0	0	0	0	0	0	0	1·0	0·6	0·1	0	0	0	0
NO	0	0	0	0	0·1	0·6	1·0	0	0	0	0	0	0	0
NS	0	0	0·1	0·5	1·0	0·8	0·3	0	0	0	0	0	0	0
NM	0·2	0·7	1·0	0·7	0·2	0	0	0	0	0	0	0	0	0
NB	1·0	0·8	0·4	0·1	0	0	0	0	0	0	0	0	0	0

Fig. 6.15. Fuzzy sets for the variables PE (Pressure Error) and SE (Speed Error)

The variables CPE and CSE have been similarly quantized; the division of zero was of course omitted here. The subjective fuzzy sets were formed as follows (see Fig. 6.16):

	−6	−5	−4	−3	−2	−1	0	+1	+2	+3	+4	+5	+6
PB	0	0	0	0	0	0	0	0	0	0·1	0·4	0·8	1·0
PM	0	0	0	0	0	0	0	0	0·2	0·7	1·0	0·7	0·2
PS	0	0	0	0	0	0	0	0·9	1·0	0·7	0·2	0	0
NO	0	0	0	0	0	0·5	1·0	0·5	0	0	0	0	0
NS	0	0	0·2	0·7	1·0	0·9	0	0	0	0	0	0	0
NM	0·2	0·7	1·0	0·7	0·2	0	0	0	0	0	0	0	0
NB	1·0	0·8	0·4	0·1	0	0	0	0	0	0	0	0	0

Fig. 6.16. Fuzzy sets for the variables CPE (Change in Pressure Error) and CSE (Change in Speed Error)

The variable HC was ultimately quantized over 15 points. The subjective fuzzy sets are the follows (see Fig. 6.17):

	−7	−6	−5	−4	−3	−2	−1	0	+1	+2	+3	+4	+5	+6	+7
PB	0	0	0	0	0	0	0	0	0	0	0	0·1	0·4	0·8	1·0
PM	0	0	0	0	0	0	0	0	0	0·2	0·7	1·0	0·7	0·2	0
PS	0	0	0	0	0	0	0	0·4	1·0	0·8	0·4	0·1	0	0	0
NO	0	0	0	0	0	0	0·2	1·0	0·2	0	0	0	0	0	0
NS	0	0	0	0·1	0·4	0·8	1·0	0·4	0	0	0	0	0	0	0
NM	0	0·2	0·7	1·0	0·7	0·2	0	0	0	0	0	0	0	0	0
NB	1·0	0·8	0·4	0·1	0	0	0	0	0	0	0	0	0	0	0

Fig. 6.17. Fuzzy sets for the variable HC (Heat Change)

Similarly, the variable TC was distributed over five points (see Fig. 6.18): Mamdani and Assilian now defined 24 rules as IF-THEN rules, such as:

$$\text{IF } PE \text{ is } NB \text{ THEN } HC \text{ is } PB,$$

	−2	−1	0	+1	+2
PB	0	0	0	0·5	1·0
PS	0	0	0·5	1·0	0·5
NO	0	0·5	1·0	0·5	0
NS	0·5	1·0	0·5	0	0
NB	1·0	0·5	0	0	0

Fig. 6.18. Fuzzy sets for the variable TC (Throttle Change)

or to issue three rules according to which the rest of the process should occur[141]:

Rule 1: IF PK and N, THEN NK.

(If the deviation in pressure is small and positive and the deviation in pressure does not change much, then reduce the supply of heat a little.)

Rule 2: IF PO and N, THEN N.

(If the deviation in pressure is approximately zero and the deviation in pressure does not change much, then do not change the supply of heat.)

Rule 3: IF PK and PK, THEN NK.

(If the deviation in pressure is small and positive and the deviation in pressure is slowly increasing, then reduce the supply of heat a little.)

These rule relationships were implemented as fuzzy relations for which Zadeh had already indicated the max-min rule as a composition rule in his first publication on Fuzzy Sets.[142] Additionally, a PDP 8/S digital computer[143] calculated a corresponding fuzzy set as a value for the output variable (Fig. 6.13). This method can be represented graphically in the following way (see Fig. 6.19): The sensors indicate sharp values for the input variables *pressure deviation* and its *change*, whose membership values with respect to the corresponding fuzzy sets can be read on the triangular membership functions. In the illustrated example for rule 1, the membership value with respect to the fuzzy set *pressure deviation* PK is 0.2 and it is 0. 4 with respect to the fuzzy set change in *pressure deviation* N. Today this part of the fuzzy control process is known as "fuzzification".

The max-min rule prescribes that the maximum of these two values is computed first. (In the example for rule 1 illustrated above, this value is 0.2).

[141] The representation of these control examples stems from [370] McNeill et al., Fuzzy (1993). For the sake of simplicity, the authors of that work did not differentiate between "positive zero" and "negative zero"
[142] See Chap. 5
[143] [26] Assilian, Control (1974), p. 17

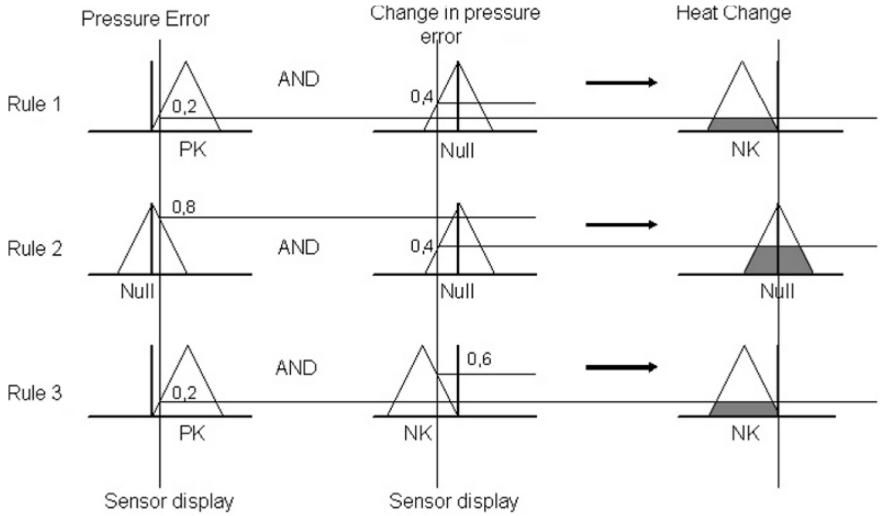

Fig. 6.19. Illustration of the application of the min-max rule

Accordingly, after executing this rule model alone, the output command was *change heat supply* NK and it had a membership value of 0.2! The result of rule 1 thus results in a triangular function that is truncated at the value 0.2 – a trapezoidal membership function.
However, rule 2 and rule 3 have also fired and so they must be evaluated analogously and parallel to rule 1. The final membership function for the fuzzy set as a value of the output variable *change in pressure deviation* is ultimately composed of the trapezoidal membership functions of the individual rule results. This composition occurs according to the max-min rule by forming the maximum of the membership functions of all three output fuzzy sets.

> To recapitulate, two algorithms were implemented in this application: one to compute the "heat change" (HC) control action and the other to compute the "throttle change" (TC) control action. Every rule in these algorithms is a relationship between the input variables PE, CPE, SE, CSE (in that order) and either HC or TC. The control actions are computed by presenting values for the input variables to the two algorithms. The input vectors are of course obtained by sampling the states of the steam engine at the sampling instants. The output of either algorithm is obviously a fuzzy set which assigns grades (of membership) to the possible values of the control fuzzy variable. In order to take a deterministic action one of those values must be chosen, the choice procedure depending on the grades of membership.[144]

[144] [348] Mamdani, Fuzzy (1975), p. 5

IF PE = (NB OR NM) THEN IF CPE = NS THEN HC = PM
OR
IF PE = NS THEN IF CPE = PS THEN HC = PM
OR
IF PE = NO THEN IF CPE = (PB OR PM) THEN HC = PM
OR
IF PE = NO THEN IF CPE = (NB OR NM) THEN HC = NM
OR
IF PE = PO OR NO THEN IF CPE = NO THEN HC = NO
OR
IF PE = PO THEN IF CPE = (NB OR NM) THEN HC = PM
OR
IF PE = PO THEN IF CPE = (PB OR PM) THEN HC = NM
OR
IF PE = PS THEN IF CPE = (PS OR NO) THEN HC = NM
OR
IF PE = (PB OR PM) THEN IF CPE = NS THEN HC = NM

Fig. 6.20. Fuzzy control commands for the steam engine designed by Assilian and Mamdani. Abbreviations used for the linguistic values: ZE, zero; PZ, positive zero; PS, positive small; PM, positive medium; PB, positive big; similar for the negative values NZ, NS, NM and NB. Negative deviations signify a movement toward the set point, positive deviations signify a movement away from the set point. Other abbreviations: PE, pressure error; CPE, change in pressure error; HC, heat input change

Just how was the output variable change in pressure deviation supposed to be adjusted, though? For this a sharp value is required. Mamdani and Assilian discussed this question only briefly in their article. For this aspect of the fuzzy control process, which is called "defuzzification", a number of different "defuzzification methods" have been proposed in the intervening years. As this development was beginning, Mamdani and Assilian decided on a simple procedure:

> Various considerations may influence the choice procedure depending on the particular application and in our case effectively that action is taken which has the largest membership grade. It is possible of course that more than one peak of a flat is obtained as illustrated below [see Fig. 6.21]. The particular procedure in our case takes the action indicated by the arrow, which is midway between the two peaks or at the centre of the plateau.[145]

In the dissertation entitled *Artificial Intelligence in the Control of Real Dynamic Systems* that Assilian produced in response to this fuzzy control problem, he wrote that the control strategy they had realized was one that a human operator – in this case, Assilian himself – could use to control a steam engine.

> These control policies were established first by imagining the entire state space ($PE \times CPE \times SE \times CSE$) to be divided into a number of areas, and second, writing down a control policy for each of these

[145] [348] Mamdani, Fuzzy (1975), p. 5; Fig. 1, ibid

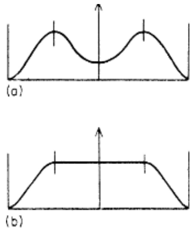

Fig. 6.21. Illustration of the selection of the centroid as a defuzzification method by devised Assilian and Mamdani

> areas. Obviously, the first set of rules obtained in this manner does not necessarily produce the best quality of control possible ...[146]

This control algorithm was thus profoundly subjective. Not only the algorithm but also the membership function had been designed subjectively. Yet as Assilian and Mamdani managed to demonstrate, this fuzzy control system exceeded the performance of conventional control systems in several ways (see Fig. 6.22).

- Much less information is required for fuzzy control than for conventional control.
- The verbal knowledge of human experts did not have to be mathematically exact in order to be reprocessed by the automatic control.

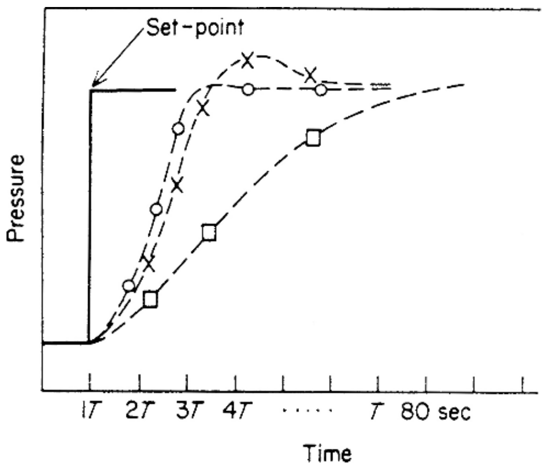

Fig. 6.22. The result of the Assilian-Mamdani Fuzzy Controller (○) compared to a conventional controller. (Dynamic Divergence Caching (DDC) algorithm damped (□) and undamped (x).)

[146] [26] Assilian, Control (1974), p. 135

- Errors were reduced little by little until the set point could be reached; digital controllers "overshot" this target instead.
- The fuzzy control system worked faster than a conventional control system; the possibility of processing the parallel firing of several rules at the same time shortened the required control time.

Therefore, the first technically realized fuzzy controller behaved exactly how Chang and Zadeh had predicted a year earlier.

Fig. 6.23. Ebrahim Mamdani (second from left), Lotfi Zadeh (fourth from left) and several of Mamdani's students in London in the late 1970s

With this fuzzy control of a steam engine – or more precisely a combination of a boiler and a steam engine – the essential principles for the construction of

an entire class of fuzzy control systems were established. Today these systems are known as "Mamdani controllers".[147]

Several years later, a first *commercial* fuzzy control system was built according to this model. Lauritz Peter Holmblad, a Danish engineer who joined F. J. Smidth & Co. upon graduation from the Technical University of Copenhagen in 1973, and Jens-Jorgen Østergaard developed a fuzzy control system for the automatic control of a cement kiln.[148]
Attempts to automate cement production had always failed in the past because the process of cement burning is highly complex, ovens do not behave linearly and only a few measurements can be taken during the process. The fuzzy cement kiln Holmblad and Østergaard developed functioned very successfully and reliably, however. It was the first large-scale commercial fuzzy system. Many other products would follow it in the 1980s. Since then, fuzzy systems have been designed, produced, marketed and sold in almost all areas of technology – first in Japan, later in the US and Europe. To describe these developments of fuzzy technology here would go beyond the bounds of this work. This is why the period covered in this book ends before the so-called "fuzzy boom" began in earnest. The fuzzy systems in the field of medicine, which this work will next address, developed in parallel with the fuzzy control systems. The final chapter is dedicated to their story.

[147] In light of the fact that Sedrak Assilian had designed this fuzzy controller and that it had also been part of his dissertation, this designation is unjust. Lotfi Zadeh put it this way: "Assilian never got any credit. ... It is a little bit unfortunate because, after all, that was his Ph.D. thesis. So this particular point requires some historical correction." [709] R. S. interview with L. A. Zadeh (2001)

[148] See: [245] Holmblad et al., Fuzzy (1982)